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Asymptotic Behavior of Almost-Orbits of Reversible Semigroups of Lipschitzian Mappings

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Let C be a nonempty closed convex subset of a Hilbert space H, G a right reversible semitopological semigroup, and $S=\{S(t): t\in G\}$ a continuous representation of G as Lipschitzian mappings of C into itself. Then we deal with the asymptotic behavior of an almost-orbit $\{u(t): t\in G\}$ of $S=\{S(t): t\in G\}$. We first prove that if $\limsup_t k_t \leqslant 1$, then the closed convex subset

$$\bigcap_{s\in G} \overline{\mathrm{co}}\{u(t): t\geqslant s\} \cap F(S)$$

consists of at most one point, where k_t is the Lipschitzian constant of $S(t)$, $t\in G$. Then we apply this result to study the problem of weak convergence of the net $\{u(t): t\in G\}$.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a mapping of C into itself. T is said to be a Lipschitzian mapping if for each $n\geqslant 1$ there exists a positive real number k_n such that

$$|T^n x - T^n y| \leqslant k_n\,|x-y|$$

for all x, $y\in C$. A Lipschitzian mapping is said to be nonexpansive if $k_n=1$ for all $n\geqslant 1$ and asymptotically nonexpansive if $\lim_n k_n = 1$, respectively. Let $S=\{S(t): t\geqslant 0\}$ be a family of nonexpansive mappings of C into itself such that $S(0)=I$, $S(t+s)=S(t)\,S(s)$ for all t, $s\in[0,\infty)$ and $S(t)x$ is continuous in $t\in[0,\infty)$ for each $x\in C$. The S is said to be a nonexpansive semigroup on C. Recently, Miyadera and Kobayasi [7] introduced the notion of an almost-orbit of a nonexpansive semigroup on C and established the weak and strong convergence of such an almost-orbit. See also [1] for an almost-orbit of a nonexpansive mapping.

In this paper, we introduce the notion of an almost-orbit $\{u(t): t\in G\}$ of

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a family $S=\{S(t): t\in G\}$ of Lipschitzian mappings of C into itself, where G is a right reversible semitopological semigroup, and prove that if $S=\{S(t): t\in G\}$ is a family of Lipschitzian mappings with $\limsup_t k_t \leqslant 1$, where k_t is the Lipschitzian constant of $S(t)$, $t\in G$, then the set

$$\bigcap_{s\in G} \overline{\text{co}}\{u(t): t\geqslant s\} \cap F(S)$$

consists of at most one point, where $\overline{\text{co}}\{u(t): t\geqslant s\}$ is the closed convex hull of $\{u(t): t\geqslant s\}$ and $F(S)$ is the set of common fixed points of $S=\{S(t): t\in G\}$. This result is applied to study the problem of weak convergence of the net $\{u(t): t\in G\}$. We also prove that if Q is the metric projection of H onto $F(S)$, then the strong limit of the net $\{Qu(t): t\in G\}$ exists. Our proofs employ the methods of Takahashi [10], Takahashi and Park [11], Ishihara and Takahashi [5], and Miyadera and Kobayasi [7]. The results are generalizations of [5, 8].

2. Definitions and Lemmas

Let G be a semitopological semigroup; i.e., G is a semigroup with a Hausdorff topology such that for each $a\in G$ the mappings $g\to a\cdot g$ and $g\to g\cdot a$ from G to G are continuous. Let C be a nonempty closed convex subset of a Hilbert space H.

Definition 1. A family $S=\{S(t): t\in G\}$ of mappings from C into itself is said to be a *(continuous) representation* of G on C if S satisfies the following:

(1) $S(ts)x=S(t)\,S(s)x$ for all $t, s\in G$ and $x\in C$;

(2) For every $x\in C$, the mapping $s\to S(s)x$ from G into C is continuous.

Definition 2. Let $S=\{S(t): t\in G\}$ be a representation of G on C. S is said to be *Lipschitzian* on C if for each $t\in G$, there exists $k_t>0$ such that $|S(t)x-S(t)\,y|\leqslant k_t\,|x-y|$ for all $x, y\in C$.

A semitopological semigroup G is right reversible if any two closed left ideals of G have nonvoid intersection. If G is right reversible, $(G, \leqslant)$ is a directed system when the binary relation "$\leqslant$" on G is defined by $a\leqslant b$ if and only if $\{a\}\cup\overline{Ga}\supseteq\{b\}\cup\overline{Gb}$.

Definition 3. Let G be a right reversible semitopological semigroup and let $S = \{S(t): t \in G\}$ be a representation of G on C. A function $u: G \to C$ is said to be an *almost-orbit* of S if

$$\lim_t (\sup_s |u(st) - S(s)\, u(t)|) = 0.$$

Let $F(S)$ denote the set of all common fixed points of mappings $S(t)$, $t \in G$ in C; i.e., $F(S) = \{x \in C: S(t)x = x \text{ for all } t \in G\}$. Then the following result is proved by Ishihara and Takahashi [5].

LEMMA 1. *Let G be right reversible and let $S = \{S(t): t \in G\}$ be Lipschitzian on C with* $\limsup_t k_t \leqslant 1$. *Then $F(S)$ is closed and convex.*

We also prove the following lemmas.

LEMMA 2. *Let G be right reversible and let $S = \{S(t): t \in G\}$ be Lipschitzian on C with* $\limsup_t k_t \leqslant 1$. *If $\{u(t): t \in G\}$ and $\{v(t): t \in G\}$ are almost-orbits of $S = \{S(t): t \in G\}$, then the limit of $|u(t) - v(t)|$ exists. In particular, for every $z \in F(S)$, the limit of $|u(t) - z|$ exists.*

Proof. Put

$$\phi(s) = \sup_t |u(ts) - S(t)\, u(s)|, \qquad \psi(s) = \sup_t |v(ts) - S(t)\, v(s)|$$

for $s \in G$. Then $\lim_s \phi(s) = \lim_s \psi(s) = 0$. Since, for any $s, t \in G$,

$$\begin{aligned} |u(ts) - v(ts)| &\leqslant |u(ts) - S(t)\, u(s)| + |S(t)\, u(s) - S(t)\, v(s)| \\ &\quad + |S(t)\, v(s) - v(ts)| \\ &\leqslant \phi(s) + \psi(s) + k_t\, |u(s) - v(s)|, \end{aligned}$$

we have

$$\begin{aligned} \inf_t \sup_{t \leqslant \tau} |u(\tau) - v(\tau)| &\leqslant \phi(s) + \psi(s) + (\inf_t \sup_{t \leqslant \tau} k_\tau)\, |u(s) - v(s)| \\ &\leqslant \phi(s) + \psi(s) + |u(s) - v(s)|, \end{aligned}$$

and then

$$\inf_t \sup_{t \leqslant \tau} |u(\tau) - v(\tau)| \leqslant \sup_t \inf_{t \leqslant s} |u(s) - v(s)|.$$

Thus, $\lim_t |u(t) - v(t)|$ exists. Let $z \in F(S)$ and put $v(t) \equiv z$. Then $v(t)$ is an almost-orbit and the limit of $|u(t) - z|$ exists.

LEMMA 3. *Let G be right reversible and let $S=\{S(t): t\in G\}$ be Lipschitzian on C with* $\limsup_t k_t \leqslant 1$. *Let $\{u(t): t\in G\}$ be an almost-orbit of $S=\{S(t): t\in G\}$. If $F(S)\neq\varnothing$, then there exists $t_0\in G$ such that $\{u(t): t\geqslant t_0\}$ is bounded.*

Proof. Let $z\in F(S)$. Then, since $\lim_t |u(t)-z|$ exists by Lemma 2, there is $t_0\in G$ such that $\{|u(t)-z|: t\geqslant t_0\}$ is bounded. Hence $\{u(t): t\geqslant t_0\}$ is bounded.

Let $\{x_\alpha\}$ be a bounded net of a Hilbert space H and let C be a nonempty closed convex subset of H. Then we define

$$r(x)=r(\{x_\alpha\}, x)=\limsup_\alpha |x_\alpha - x|$$

and

$$r=r(\{x_\alpha\})=\inf\{r(x): x\in C\}.$$

It is well known that there exists a unique point $a\in C$ such that $r(a)=r$. The point a is called the *asymptotic center* of $\{x_\alpha\}$ in C, and is denoted by $\mathrm{AC}(\{x_\alpha\})$.

LEMMA 4 (Lim [6]). *Let $\{y_\beta\}$ be a net of C such that* $\limsup_\beta r(\{x_\alpha\}, y_\beta)\leqslant r(\{x_\alpha\})$. *Then $y_\beta\to a=\mathrm{AC}(\{x_\alpha\})$.*

We also know that if $\{x_\alpha\}\subset C$ and if $\{x_\alpha\}$ converges weakly to $y\in C$, then y is the asymptotic center of $\{x_\alpha\}$ in C [2, Theorem 4.2]. Let Q be the metric projection of H onto $F(S)$. Then, by Phelps [9], Q is nonexpansive.

LEMMA 5. *Let G be right reversible and let $S=\{S(t): t\in G\}$ be Lipschitzian on C with* $\limsup_t k_t\leqslant 1$. *Suppose that $F(S)\neq\varnothing$. If $\{u(t): t\in G\}$ is an almost-orbit of $S=\{S(t): t\in G\}$, then $\{Qu(t): t\in G\}$ converges strongly to the asymptotic center of $\{u(t): t\in G\}$ in $F(S)$.*

Proof. Let z be the asymptotic center of $\{u(t): t\in G\}$ in $F(S)$. Then, for all $s, t\in G$, we have

$$\begin{aligned} r(Qu(s)) &\leqslant \sup_{a\geqslant ts} |u(a)-Qu(s)| = \sup_{a\geqslant t} |u(as)-Qu(s)| \\ &\leqslant \sup_{a\geqslant t} |u(as)-S(a)\,u(s)| + \sup_{a\geqslant t} |S(a)\,u(s)-Qu(s)| \\ &\leqslant \phi(s) + (\sup_{a\geqslant t} k_a)\,|u(s)-Qu(s)| \leqslant \phi(s)+(\sup_{a\geqslant t} k_a)\,|u(s)-z|, \end{aligned}$$

where $\phi(s) = \sup_t |u(ts) - S(t)\, u(s)|$. Since $\lim \phi(s) = 0$, it follows that

$$r(Qu(s)) \leqslant \phi(s) + (\limsup_t k_t)\, |u(s) - z| \leqslant \phi(s) + |u(s) - z|$$

and

$$\limsup_s r(Qu(s)) \leqslant \limsup_s |u(s) - z| = r.$$

By Lemma 4, we obtain $Qu(t) \to z$.

3. Asymptotic Behavior of Almost-Orbits

In this section, we study the asymptotic behavior of an almost-orbit $\{u(t): t \in G\}$ of $S = \{S(t): t \in G\}$.

THEOREM 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a right reversible semitopological semigroup and let $S = \{S(t): t \in G\}$ be a Lipschitzian representation of G on C with* $\limsup_t k_t \leqslant 1$. *Let $\{u(t): t \in G\}$ be an almost-orbit of $S = \{S(t): t \in G\}$. Suppose that $F(S) \neq \varnothing$. Then the set*

$$\bigcap_t \overline{\text{co}}\{u(s): s \geqslant t\} \cap F(S)$$

consists of at most one point.

Proof. Let Q be the metric projection of H onto $F(S)$. Then $\{Qu(t): t \in G\}$ converges strongly to the asymptotic center z of $\{u(t): t \in G\}$ in $F(S)$ by Lemma 5. Now, let $u \in \bigcap_t \overline{\text{co}}\{u(s): s \geqslant t\} \cap F(S)$. By Lemma 2, the limits of $|u(t) - z|$ and $|u(t) - u|$ exist. Since, for each $s \in G$,

$$|z - u|^2 = |u(s) - u|^2 - |u(s) - z|^2 - 2\langle z - u, u(s) - z\rangle,$$

it follows that

$$\begin{aligned} |z - u|^2 + 2 \lim_s \langle z - u, u(s) - z\rangle &= \lim_s |u(t) - u|^2 - \lim_s |u(t) - z|^2 \\ &= \limsup_t |u(t) - u|^2 - \limsup_t |u(t) - z|^2 \\ &= r(u)^2 - r^2 \geqslant 0. \end{aligned}$$

Let $\varepsilon > 0$. Then we have

$$2 \lim \langle z - u, u(s) - z\rangle \geqslant -|z - u|^2 > -|z - u|^2 - \varepsilon,$$

and hence there exists $s_0 \in G$ such that

$$2\langle z-u, u(s)-z\rangle > -|z-u|^2 - \varepsilon$$

for all $s \geqslant s_0$. Since $u \in \overline{\mathrm{co}}\{u(t): t \geqslant s_0\}$, we have

$$2\langle z-u, u-z\rangle \geqslant -|z-u|^2 - \varepsilon,$$

which implies that $|z-u|^2 \leqslant \varepsilon$. Since ε is arbitrary, we have $z=u$. Therefore,

$$\bigcap_s \overline{\mathrm{co}}\{u(t): t \geqslant s\} \cap F(S) = \{z\}.$$

By using Theorem 1, we study the problem of the weak convergence of $\{u(t): t \in G\}$. For a function $u: G \to C$, let $\omega(u)$ denote the set of all weak limit points of subnets of the net $\{u(t): t \in G\}$. If $\{u(t): t \in G\}$ is an almost-orbit of $S = \{S(t): t \in G\}$ and $F(S) \neq \varnothing$, then $\{u(t): t \geqslant t_0\}$ is bounded for some $t_0 \in G$, and hence $\omega(u) \neq \varnothing$.

THEOREM 2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a right reversible semitopological semigroup and let $S = \{S(t): t \in G\}$ be a Lipschitzian representation of G on C with* $\limsup_t k_t \leqslant 1$. *Suppose that $F(S) \neq \varnothing$. Let $\{u(t): t \in G\}$ be an almost-orbit of $S = \{S(t): t \in G\}$. If $\omega(u) \subset F(S)$, then the net $\{u(t): t \in G\}$ converges weakly to some $y \in F(S)$.*

Proof. By Lemma 5, the net $\{Qu(t): t \in G\}$ converges strongly to some $y \in F(S)$. Since $\omega(u) \subset \bigcap_t \overline{\mathrm{co}}\{u(s): s \geqslant t\}$, it follows that

$$\bigcap_t \overline{\mathrm{co}}\{u(s): s \geqslant t\} \cap F(S) \supset \omega(u) \neq \varnothing.$$

Hence, from Theorem 1, we have

$$\{y\} = \bigcap_t \overline{\mathrm{co}}\{u(s): s \geqslant t\} \cap F(S).$$

Therefore, $\{y\} = \omega(u)$, that is, the net $\{u(t): t \in G\}$ converges weakly to y.

By Theorem 2 and Lemma 5, we have the following:

THEOREM 3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a right reversible semitopological semigroup and let $S = \{S(t): t \in G\}$ be a Lipschitzian representation of G on C with* $\limsup_t k_t \leqslant 1$. *Suppose that $F(S) \neq \varnothing$. Let $\{u(t): t \in G\}$ be an almost-orbit of $S = \{S(t): t \in G\}$. Then $\{u(t): t \in G\}$ converges weakly to some $y \in C$ if and only* *if $u(ht)-u(t)$ converges weakly to* 0 *for all $h \in G$. In this case, $y \in F(S)$ and* $\lim_t Qu(t) = y$.

Proof. We need only prove the "if" part. By Theorem 2, it suffices to show that $\omega(u) \subset F(S)$. Let $\{u(t_\alpha)\}$ be a subnet of $\{u(t): t \in G\}$ converging weakly to $z \in C$, and let $\varepsilon > 0$. Since $\{u(t_\alpha)\}$ is bounded and $\limsup_t k_t \leqslant 1$, there exists $t_0 \in G$ such that for $t \geqslant t_0$ and any α,

$$k_t^2 \, |u(t_\alpha) - z|^2 \leqslant |u(t_\alpha) - z|^2 + \varepsilon.$$

Let $u \in F(S)$. Then we have, for $t \geqslant t_0$ and all α,

$$\begin{aligned}
-\varepsilon &\leqslant k_t^2 \, |u(t_\alpha) - z|^2 - |S(t)\, u(t_\alpha) - S(t) z|^2 - \varepsilon \\
&\leqslant |u(t_\alpha) - z|^2 - |S(t)\, u(t_\alpha) - S(t) z|^2 \\
&= |u(t_\alpha) - u|^2 + 2\langle u(t_\alpha) - u, u - z \rangle + |u - z|^2 \\
&\quad - |S(t)\, u(t_\alpha) - u|^2 - 2\langle S(t)\, u(t_\alpha) - u, u - S(t) z \rangle - |u - S(t) z|^2 \\
&= |u(t_\alpha) - u|^2 - |S(t)\, u(t_\alpha) - u|^2 + |u - z|^2 - |u - S(t) z|^2 \\
&\quad + 2\langle u(t_\alpha) - u, S(t) z - z \rangle + 2\langle u(t_\alpha) - S(t)\, u(t_\alpha), u - S(t) z \rangle.
\end{aligned}$$

Since $\{u(t): t \in G\}$ is an almost-orbit of $S = \{S(t): t \in G\}$ and $u(hs) - u(s)$ converges weakly to 0 for all $h \in G$, it is easy to verify that

$$\lim_\alpha |S(t)\, u(t_\alpha) - u|^2 = \lim_\alpha |u(t_\alpha) - u|^2,$$

$$u(t_\alpha) - S(t)\, u(t_\alpha) \to 0 \qquad \text{weakly.}$$

Thus

$$-\varepsilon \leqslant 2\langle z - u, S(t) z - z \rangle + |z - u|^2 - |u - S(t) z|^2 = -|z - S(t) z|^2$$

for $t \geqslant t_0$. It follows that $\limsup_t |S(t) z - z|^2 \leqslant \varepsilon$. Since ε is arbitrary, we have $\lim_t S(t) z = z$. Now, for $s \in G$,

$$S(s) z = \lim_t S(s)\, S(t) z = \lim_t S(st) z = z;$$

i.e., $z \in F(S)$ and hence $\omega(u) \subset F(S)$. By Theorem 2, the net $\{u(t): t \in G\}$ converges weakly to some $y \in F(S)$. Since y is the asymptotic center of $\{u(t): t \in G\}$ in $F(S)$, by Lemma 5, we obtain that $\lim_t Qu(t) = y$.

As a direct consequence, we have the following:

COROLLARY. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a right reversible semitopological semigroup and let $S =$*

$\{S(t)\colon t \in G\}$ be a Lipschitzian representation of G on C with $\limsup_t k_t \leqslant 1$. *Suppose that $F(S) \neq \emptyset$. Let $\{u(t)\colon t \in G\}$ be an almost-orbit of $S = \{S(t)\colon t \in G\}$. If* $\lim_t |u(ht) - u(t)| = 0$ *for all $h \in G$, then the net $\{u(t)\colon t \in G\}$ converges weakly to some $y \in F(S)$.*

REFERENCES

1. R. E. BRUCK, A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces, *Israel J. Math.* **32** (1979), 107–116.
2. K. GOEBEL AND S. REICH, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Dekker, New York, 1984.
3. K. GOEBEL, W. A. KIRK, AND R. L. THELE, Uniformly lipschitzian families of transformations in Banach space, *Canad. J. Math.* **26** (1974), 1245–1256.
4. H. ISHIHARA AND W. TAKAHASHI, Fixed point theorems for uniformly lipschitzian semigroups in Hilbert spaces, *J. Math. Anal. Appl.* **127** (1987), 206–210.
5. H. ISHIHARA AND W. TAKAHASHI, A nonlinear ergodic theorem for a reversible semigroup of lipschitzian mappings in a Hilbert space, *Proc. Amer. Math. Soc.* **104** (1988), 431–436.
6. T. C. LIM, On asymptotic centers and fixed points of nonexpansive mappings, *Proc. Amer. Math. Soc.* **84** (1982), 212–216.
7. I. MIYADERA AND K. KOBAYASI, On the asymptotic behavior of almost-orbits of nonlinear contraction semigroups in Banach spaces, *Nonlinear Anal.* **6** (1982), 349–365.
8. G. MOROSANU, Asymptotic behavior of solutions of differential equations associated to monotone operators, *Nonlinear Anal.* **3** (1979), 873–883.
9. R. R. PHELPS, Convex sets and nearest points, *Proc. Amer. Math. Soc.* **8** (1957), 790–797.
10. W. TAKAHASHI, A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space, *Proc. Amer. Math. Soc.* **97** (1986), 55–58.
11. W. TAKAHASHI AND J. Y. PARK, On the asymptotic behavior of almost-orbits of commutative semigroups in Banach spaces, *in* "Nonlinear and Convex Analysis," pp. 271–293, Dekker, New York/Basel, 1987.